1	2	3	4
~~24~~	8 16	8 24	24

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Sets & Numbers

Natalie [redacted]

In total: 2 sheets of paper, 8 page

1.

$X = \{1,2,3,4,5\}$

8 $Y = \{2,4,6,8\}$

$Z = \{8,-1,4\}$

a) $(X \cap Y) \setminus Z = ?$

$X \cap Y = \{1,2,3,4,5\} \cap \{2,4,6,8\} = \{2,4\}$

↳ elements in X and ~~...~~ in Y

$(X \cap Y) \setminus Z = \{2,4\} \setminus \{8,-1,4\} = \underline{\underline{\{2\}}}$ ✓

↳ elements of X and Y but not in Z

b) $f: Z \to X$ that's not injective.

8 Not injective ⇒ not all ~~elements~~ distinct elements of Z are mapped to distinct elements of X.

$f: Z \to X$
$8 \mapsto 1$
$-1 \mapsto 1$
$4 \mapsto 2$

alternatively $f(8) = 1$
$f(-1) = 1$
$f(4) = 2$

↳ well-defined, - all ~~...~~ elements have mapping

↳ $f(8) = 1 = f(-1)$, so $f(8) = f(-1)$ ~~...~~ while $8 \neq -1$,
so $f(a) = f(b) \Rightarrow a = b$ does <u>not</u> hold for $\forall a,b \in Z$ ✓

c) 8 $g: Y \to Z$
$g(2) = g(4) = 4$
$g(6) = -1$
$g(8) = 8$

$g^{-1}(\{4,-1\}) = \{y \in Y : g(y) \in \{4,-1\}\} = \underline{\underline{\{2,4,6\}}}$ ✓

inverse image

2. continues

2. X, Y arbitrary sets

a) Prove that $2^X \cap 2^Y \subseteq 2^{X \cap Y}$.

Proof: The power set 2^X ~~can~~ is a set of all subsets of X.
The power set 2^Y is a set of all subsets of Y.
Let S be an arbitrary element of $2^X \cap 2^Y$.
By the definition of intersection, $S \in 2^X$ and $S \in 2^Y$.
This means that S is a subset of X and S is a subset of Y.
S being a subset of X means (by def of a subset) that all elements of S are also elements of X, and same holds for Y (because $S \subseteq Y$).
We can take an arbitrary element of S, let's call it t. $t \in X$ and $t \in Y$ by ~~the~~ the statement of the previous sentence. Because t is in both X and Y $\Rightarrow t \in X \cap Y$. ~~Because As the~~ t is arbitrary and so this holds for all elements of S. Because all elements of S are also elements of $X \cap Y$, S is a subset of $X \cap Y$.
$2^{X \cap Y}$ is the set of all subsets of $X \cap Y$. ~~Beca~~
$S \subseteq X \cap Y \Rightarrow S \in 2^{X \cap Y}$.
As it was shown that an arbitrary element of $2^X \cap 2^Y$ is also an element of $2^{X \cap Y}$, we can conclude that $2^X \cap 2^Y$ is a subset of $2^{X \cap Y}$, hence proving the statement.

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(2.)

b) Prove that if $X \subseteq Y$, ~~and~~ eq. rel. R on Y $(R \subseteq Y^2)$, then $E = X^2 \cap R$ is an equivalence relation on X.

Proof: R is an equivalence relation on Y, so it satisfies following properties:

1. reflexivity: $\forall y \in Y$, $(y, y) \in R$.
2. symmetry: $\forall y_1, y_2 \in Y$, $(y_1, y_2) \in R \Rightarrow (y_2, y_1) \in R$.
3. transitivity: $\forall y_1, y_2, y_3 \in Y$, $(y_1, y_2) \in R$ and $(y_2, y_3) \in R \Rightarrow (y_1, y_3) \in R$.

X^2 is a set of all pairs (x_1, x_2) s.t. $x_1, x_2 \in X$, and because $X \subseteq Y$, ~~any~~ x_1, x_2 are also elements of Y.

To prove that $E = X^2 \cap R$ is an equivalence relation on X, we must show that it satisfies these three properties: reflexivity, symmetry, transitivity.

1. reflexivity: $\forall x \in X$, $(x, x) \in X^2$ by the virtue of X^2 containing all the pairs of elements of X, and $(x, x) \in R$ because $x \in Y$ and by reflexivity of R, ~~that~~ for all elements y of ~~R~~ Y, we have that $(y, y) \in R$.
Because $(x, x) \in X^2$ and $(x, x) \in R$, then $(x, x) \in X^2 \cap R = E$ $\forall x \in X$. This shows that E is reflexive. S

2. symmetry: $\forall x_1, x_2 \in X$, $(x_1, x_2) \in X^2$ and $(x_2, x_1) \in X^2$. We need to show that $(x_1, x_2) \in E \Rightarrow (x_2, x_1) \in E$. Because both pairs are in X^2, the implication will hold if presence of ~~one~~ (x_1, x_2) in R will imply presence of (x_2, x_1) in R. Because R is an equivalence relation (and $x_1, x_2 \in Y$), we get that $(x_1, x_2) \in R \Rightarrow (x_2, x_1) \in R$ by the symmetry property of R. And so the necessary implication is shown, hence E is symmetric. S

3. transitivity: $\forall x_1, x_2, x_3 \in X$, we have $(x_1, x_2) \in X^2$, $(x_2, x_3) \in X^2$, $(x_1, x_3) \in X^2$.
If $(x_1, x_2) \in E$ and $(x_2, x_3) \in E$, then $(x_1, x_2) \in R$ and $(x_2, x_3) \in R$ (and they are also in X^2). From transitivity of R, $(x_1, x_2) \in R$ and $(x_2, x_3) \in R \Rightarrow (x_1, x_3) \in R$. Because in such case $(x_1, x_3) \in R$ and $(x_1, x_3) \in X^2$ (always), we have that $(x_1, x_3) \in X^2 \cap R = E$.

2.b) continuation

Because $(x_1, x_2) \in E$ and $(x_2, x_3) \in E$, implies $(x_1, x_3) \in E$, the transitivity of E is shown. 8

~~The~~ E satisfies the 3 properties of an equivalence relation and ~~because~~ $E = X^2 \cap R \Rightarrow E \subseteq X^2$, we can conclude that E is indeed an equivalence relation on X.

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2.c) Define bijection $b: X \times Y \to Y \times X$ and prove that b is a bijection.

Let $b: X \times Y \to Y \times X$
$(x, y) \mapsto (y, x)$

Proof: b is a bijection $\Leftrightarrow$ b is injective and surjective.

1. injectivity: Take arbitrary $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ (~~[illegible]~~ (x_1, y_1), ~~and~~ $(x_2, y_2) \in X \times Y$) ~~and suppose~~ such that

* NB

$$b(x_1, y_1) = b(x_2, y_2)$$
$\Leftrightarrow (y_1, x_1) = (y_2, x_2)$ by definition of b
$\Leftrightarrow y_1 = y_2$ and $x_1 = x_2$.
$\Leftrightarrow (x_1, y_1) = (x_2, y_2)$

Because $b(x_1, y_1) = b(x_2, y_2) \Rightarrow (x_1, y_1) = (x_2, y_2)$ ~~$\forall (x_1, y_1), (x_2$~~ $\forall (x_1, y_1), (x_2, y_2) \in X \times Y$ ~~[illegible]~~, b is injective.

2. surjectivity: Let $(\tilde{y}, \tilde{x})$ be an arbitrary element of $Y \times X$. For any such element, we can find ~~its pre-image~~ $(\tilde{x}, \tilde{y}) \in X \times Y$ s.t. $b(\tilde{x}, \tilde{y}) = (\tilde{y}, \tilde{x})$; namely $(x, y) = (\tilde{x}, \tilde{y}) \in X \times Y$. So all elements of the codomain of b have a pre-image in the domain of b, hence b is surjective.

* If X or Y is the empty set, then $X \times Y = \emptyset$ and $Y \times X = \emptyset$. In such case the bijection does nothing b/c it has no elements to act on. Still, for $\forall a, b \in X \times Y$ (there aren't any)
$a \neq b \Rightarrow f(a) \neq f(b)$.

Also b is surjective because $b(\emptyset) = \emptyset$ which is the codomain ✓

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3. a) example of infinite set ~~which~~ which is ~~to~~ not countable:
We can take for example $\mathbb{R}$ = set of real numbers.
Because there does not exist a bijection between $\mathbb{R}$ and $\mathbb{N}$, it is not countable.

Another example could be the power set $2^{\mathbb{N}}$.
By Cantor's diagonal element theorem,
$\# X \neq \# 2^{X}$, so in this case $\# \mathbb{N} \neq \# 2^{\mathbb{N}}$.
Countably infinite are such sets that ~~∃ bijection between the~~ they have that same cardinality as $\mathbb{N}$. Clearly, $2^{\mathbb{N}}$ does not have the same cardinality as $\mathbb{N}$, and so it cannot be countably infinite.

b) Prove that for any set $X \neq \emptyset$ and any $x \in X$, we have $\{x\} \cup (X \setminus \{x\}) = X$.

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Proof: Two sets A and B are equal $\iff A \subseteq B$ and $B \subseteq A$.

$\subseteq$ We aim to show that $\{x\} \cup (X \setminus \{x\}) \subseteq X$.
All elements in $\{x\} \cup (X \setminus \{x\})$ are either the element x, or elements of $X \setminus \{x\}$.
$x \in X$ ✓ and all elements of $X \setminus \{x\}$ are also elements of X (they are such elements of X which are not in $\{x\}$) ✓. Therefore all elements of $\{x\} \cup (X \setminus \{x\})$ are also elements of X, hence $\{x\} \cup (X \setminus \{x\}) \subseteq X$.

2 Now we will show that $X \subseteq \{x\} \cup (X \setminus \{x\})$.
~~All elements~~ Let $s \in X$, arbitrary. Either $s = x$, in such case $s \in \{x\}$, or $s \neq x$, then $s \notin \{x\}$ and by definition of difference of sets, $s \in (X \setminus \{x\})$ because $s \in X$.

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If $s \in \{x\}$ or $s \in (X \setminus \{x\})$, we can say that it is certainly element of $\{x\} \cup (X \setminus \{x\})$.
Because this holds for arbitrary $s \in X$, all elements of X are also elements of $\{x\} \cup (X / \{x\})$, so $X \subseteq \{x\} \cup (X \setminus \{x\})$.

The inclusion has been proven both ways, therefore $X = \{x\} \cup (X \setminus \{x\})$.

6) Prove by induction: $\prod_{k=0}^{n} 4^k = 2^{n(n+1)}$ for any $n \in \mathbb{N}_0$

For induction, we first formulate our statement:

$$S_n : \prod_{k=0}^{n} 4^k = 2^{n(n+1)}$$

which we aim to prove ~~for~~ $\forall n \in \mathbb{N}_0$.

1. We start by proving this for $n = 0$:

$$S_0 : \prod_{k=0}^{0} 4^k = 4^0 = 1 = 2^0 = 2^{0 \cdot 1} = 2^{0 \cdot (0+1)} \quad \checkmark$$

So we see that S_0 holds.

2. For the induction step, we assume S_n holds for some $l \in \mathbb{N}$:

$S_l : \prod_{k=0}^{l} 4^k = 2^{l(l+1)}$ is true. We now aim to prove that S_{l+1} holds:

$$S_{l+1} : \prod_{k=0}^{l+1} 4^k = 4^{l+1} \cdot \underbrace{\prod_{k=0}^{l} 4^k}_{= 2^{l(l+1)} \text{ by our assumption}} = 4^{l+1} \cdot 2^{l(l+1)} = 2^{2(l+1)} 2^{l(l+1)} =$$

$$= 2^{2l+2} \cdot 2^{l(l+1)} = 2^{l(l+1)+2l+2} = 2^{l^2+l+2l+2} = 2^{l^2+3l+2} = 2^{(l+2)(l+1)}$$

$$= 2^{[(l+1)+1](l+1)}$$

which is precisely what we wanted to obtain. Hence it is shown that S_l being true implies S_{l+1} being true.

Therefore it is shown, by mathematical induction, that S_n holds $\forall n \in \mathbb{N}$.

4. $X = \{a,b,c\}$ $\quad f: X \to Y$ $\quad , g: Y \to Z$

$Y = \{4,5\}$ $\quad f(a) = 4$ $\quad g(4) = Y$

$Z = \{X, Y\}$ $\quad f(b) = f(c) = 5$ $\quad g(5) = X$.

a) g has an inverse because g is a bijection (and bijection $\Rightarrow$ existence of inverse).

To ~~prove~~ show that it is a bijection, we check whether g is injective and surjective.

1. injective: We can see this from the definition of g,

$$\begin{array}{c} 4 \to Y \\ 5 \to X \\ g \end{array}$$

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we can also say that $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$ because, concretely:

$g(4) = Y = g(4) \Rightarrow 4 = 4$ and nothing else and similarly

$g(5) = X = g(5) \Rightarrow 5 = 5$.

Or that $\forall y \in Y$, $y_1 \neq y_2 \Rightarrow g(y_1) \neq g(y_2)$

$4 \neq 5 \Rightarrow g(4) = Y \neq X = g(5)$.

2. surjective: from definition of g, we have that $\underbrace{g(Y)}_{\text{image of } Y \text{ under } g} = g(\{4,5\}) = \{X, Y\} = Z$

$\Rightarrow$ range of g is its codomain $\Rightarrow$ g is surjective.

$g^{-1}(X) = 5$

$g^{-1}(Y) = 4$

, where g^{-1} is the inverse function of g, not the inverse image.

b) Is $g \circ f$ surjective? Yes

$g \circ f: X \to Y \to Z = X \to Z$

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$g(f(a)) = g(4) = Y$

$g(f(b)) = g(5) = X$

$g(f(c)) = g(5) = X$

$g(f(X)) = \{g(f(a))\} \cup \{g(f(b))\} \cup \{g(f(c))\} = \{Y\} \cup \{X\} \cup \{X\} = \{X, Y\} = Z \Rightarrow$ T

3.b) continuation:

Because $g(f(X))$ = Image of domain = range is equal to Z = the codomain, we conclude that $g \circ f$ is surjective.

4.c) System of representants for the equivalence relation $\sim_f$ on X.

$\sim_f = \{ \cancel{\quad} (x_1, x_2) \in X \ : f(x_1) = f(x_2) \}$

X has three elements: ~~the~~ a, b, c.

$f(a) = 4$

$f(b) = 5 = f(c)$

We can see that $f(b) = f(c) \Rightarrow b \sim_f c$

and $f(a) \neq f(b)$ and $f(a) \neq f(c)$, so they are not equivalent.

Of course, in $\sim_f$, we must also include $c \sim_f b$ from symmetry and $a \sim_f a$, $b \sim_f b$, $c \sim_f c$ from reflexivity.

$\Rightarrow \sim_f = \{ (a,a), (b,b), (c,c), (b,c), (c,b) \}$

System of representants is then the set containing all ~~all~~ elements s.t. each represents a distinct equivalence class and ~~union~~ the union of the equivalence classes gives all of $\sim_f$ (or $X \backslash \sim_f$).

For $\sim_f$, ~~they~~ the system of representants = $\{a, b\}$. ✓

To show that it is indeed a system of representants, notice that

$[a] = \{(a,a)\}$

$[b] = \{(b,b), (b,c), (c,b), (c,c)\}$ 8/8

$\Rightarrow [a] \cup [b] = \sim_f$ ✓

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